On Societies Choosing Social Outcomes, and their Memberships*

Gustavo Bergantiños[†], Jordi Massó[‡] and Alejandro Neme[§]

May 28, 2015

Abstract: We consider a society whose members have to choose not only an outcome from a given set of outcomes but also the subset of agents that will remain members of the society. We characterize the class of all strategy-proof, unanimous and nonbossy rules as the family of all serial dictator rules. Moreover, we study the extensions of approval voting, plurality voting, Borda methods and Condorcet winners to our setting from the point of view of their consistency and internal stability properties.

Journal of Economic Literature Classification Number: D71.

Keywords: Strategy-proofness; Consistency; Internal Stability; Unanimity; Nonbossiness.

*The work of G. Bergantiños is partially supported by research grant ECO2011-23460 from the Spanish Ministry of Science and Innovation and FEDER. J. Massó acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the Severo Ochoa Programme for Centres of Excellence in R&D (SEV-2011-0075) and through grant ECO2008-0475-FEDER (Grupo Consolidado-C), and from the Generalitat de Catalunya, through the prize "ICREA Academia" for excellence in research and grant SGR2014-515. The work of A. Neme is partially supported by the Universidad Nacional de San Luis, through grant 319502, and by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), through grant PIP 112-200801-00655.

[†]Research Group in Economic Analysis. Facultade de Económicas, Universidade de Vigo. 36310, Vigo (Pontevedra), Spain. E-mail: gbergant@uvigo.es

[‡]Universitat Autònoma de Barcelona and Barcelona Graduate School of Economics. Departament d'Economia i d'Història Econòmica, Campus UAB, Edifici B. 08193, Bellaterra (Barcelona), Spain. E-mail: jordi.masso@uab.es

[§]Instituto de Matemática Aplicada de San Luis. Universidad Nacional de San Luis and CONICET. Ejército de los Andes 950. 5700, San Luis, Argentina. E-mail: aneme@unsl.edu.ar

1 Introduction

A classical social choice problem is the following. A society N of agents has to choose an outcome from a given set X. Since agents may have different preferences over X, and it is desirable that the chosen outcome be perceived as a compromise among the potentially different preferences, agents have to be asked about them. A social choice function (a rule) collects individual preferences and selects in a systematic way an outcome, taking into account the revealed preference profile (the list of individual preferences, one for every agent).

This classical approach assumes that the composition of the society is independent of the chosen outcome. There are many situations for which this assumption is not appropriate because the composition of the society may depend on the chosen outcome. For instance, membership of a political party may depend on the positions that the party takes on issues like the death penalty, abortion, or the possibility of allowing the independence of a region of the country. A professor in a department may consider to look for a position in another university if he considers that the recruitment of the department has not being satisfactory to his standards. Hence, to be able to deal with such situations the classical social choice model has to be modified to include explicitly the possibility that members may leave the society as the consequence of the chosen outcome.

There is a literature that has already considered explicitly the dependence of the final society on its choices in specific settings. For instance, Barberà, Mashler and Shalev (2001) consider a dynamic setting in which the sets of founders and candidates are fixed, and the society holds elections for a fixed number of periods using voting by quota 1 (one vote is sufficient for admission, and voters can support as many candidates as they wish). They show that very interesting strategic behavior may emerge in equilibrium, even when the used voting method is very simple. Giving the right to vote to elected candidates and not allowing non elected candidates to vote at all, are two extreme ways of transferring influence among agents. Barberà and Perea (2002) study a similar model in which the transfer of influence to new members or non elected candidates behaves in a continuous way instead of being binary. They study the (essentially) unique subgame perfect equilibrium of a model with these features and identify its simple dynamic structure. Berga, Bergantiños, Massó and Neme (2004) study also the problem of a society choosing a subset of new members, from a finite set of candidates, using voting by committees as in Barberà, Sonnenschein and

Zhou (1991). They consider explicitly the possibility that initial members of the society (founders) may want to exit, if they do not like the resulting new society. They show that, if founders have separable (or additive) preferences, the unique strategy-proof, stable and onto rule is the one where candidates are chosen unanimously and no founder exits. Berga, Bergantiños, Massó and Neme (2006) study equilibria of a finite extensive form game in which, after knowing the chosen alternative, members may reconsider their membership by either staying or exiting. In turn, and as a consequence of the exit of some of its members, other members might now find undesirable to belong to the society as well. For general exit procedures they analyze the exit behavior of members after knowing the chosen alternative. All these papers mentioned above study specific models in terms of the voting methods under which members choose the social outcome and the timing under which members reconsider their membership.

In most part of this paper we look at the general setting without being specific about the two issues. We do that by considering that the set of alternatives are all pairs formed by a subset of the original society (an element in 2^N, the subset of agents that will remain in the society) and an outcome in X. Then, we assume that agents' preferences are defined over the set $2^N \times X$ of alternatives and satisfy two natural requirements. First, each agent has strict preferences between any two alternatives, provided the agent belongs to the two corresponding societies. Second, each agent is indifferent between two alternatives, provided the agent is not a member of any of the two corresponding societies; namely, agents do not care about the outcome chosen by societies they do not belong to.

We consider rules that operate on this restricted domain of profiles by selecting, for each profile, an alternative (a final society and an outcome). An agent that understands the effect of the revealed preference on the selected alternative faces an strategic problem: how to select the best revealed preference. Depending on the rule under consideration, the agent may realize that the solution to this problem is ambiguous because it may depend on the agent's expectations that he has about the revealed preferences of the other agents, and in turn he may also realize that to formulate hypothesis about those revealed preferences require hypothesis about the others' expectations, and so on. Strategy-proof rules make all these considerations unnecessary since truthtelling is a weakly dominant strategy of the direct revelation game form at each profile; namely, each agent's decision problem is independent of the preferences revealed by the other agents. Our first result, Theorem 1, characterizes the class of all strategy-proof, unanimous and nonbossy rules as the family of all serial dictator rules. A rule is unanimous if it always selects an alternative belonging to the set of common best alternatives, whenever this set is nonempty. A rule is nonbossy if it is invariant with respect to the change of preferences of an agent who is not a member of the two final societies. A serial dictator rule, relative to an ordering of the agents, gives to the first agent the power to select his best alternative, and only if this agent has many indifferent alternatives at the top of his preference then, the second agent in the order has the power to select his best alternative among those declared as being indifferent by the first agent, and proceeds similarly following the ordering of the agents.

For applications where the profile is common knowledge (and hence, the revelation of agents' preferences is not an strategic issue) we focus on the consistency of rules (see Thomson (1994, 2007) and Bergantiños, Massó and Neme (2015) for the study of consistent rules in other social choice settings). A rule is consistent if the following property holds. Apply the rule to a given profile and consider the new problem where the new society is formed by the chosen subset of agents at the original profile. A consistent rule chooses at the subprofile of preferences of the agents that remain members of the society the same alternative (the same subset of agents and the same outcome). Thus, a consistent rule does not require to reapply the rule after an alternative has been chosen. We adapt well-known voting methods to our setting, with the goal of making them internally stable (all agents that are chosen to be members of the final society want to stay). We show that plurality voting and the Borda method do not satisfy consistency. However, approval voting not only satisfies consistency but it also satisfies other desirable properties. Finally, we show that the Condorcet winner is consistent at those profiles where an alternative beats all other alternatives by majority voting.

The paper is organized as follows. In Section 2 we describe the model. Section 3 contains the definitions of the properties of rules that we will be interested in. In Section 4 we focus on strategy-proof rules and state, as Theorem 1, the characterization of the class of all strategy-proof, unanimous and nonbossy rules as the family of all serial dictator rules. Section 5 contains the analysis of well-known rules from the point of view of their consistency and internal stability properties. An Appendix at the end of the paper contains the proof of Theorem 1.

2 Preliminaries

Let $N = \{1, ..., n\}$ be the set of *agents*, where $n \geq 2$, and let X be the set of possible *outcomes*. We are interested in situations where some agents may not be part of the final society, perhaps as the consequence of the chosen outcome. To model such situations, let $A = 2^N \times X$ be the set of (social and final) *alternatives* and assume that each agent $i \in N$ has preferences over the set of possible alternatives A. Let R_i denote agent i's (weak) *preference* over A, where for any pair of alternatives $(S, x), (T, y) \in A$, $(S, x)R_i(T, y)$ means that agent i considers alternative (S, x) to be at least as good as alternative (T, y). Let P_i and I_i denote the strict and indifference relations induced by R_i over A, respectively; namely, for any pair of alternatives $(S, x), (T, y) \in A$, $(S, x)P_i(T, y)$ if and only if $(S, x)R_i(T, y)$ and $\neg (T, y)R_i(S, x)$, and $(S, x)I_i(T, y)$ if and only if $(S, x)R_i(T, y)$ and $(T, y)R_i(S, x)$. We assume that agents do not care about the outcome chosen by a society that they do not belong to and are not indifferent between pairs of alternatives for which they are members of at least one of the two societies. Namely, we assume that agent i's preferences R_i over A satisfy the following two properties: for all $x, y \in X$ and $S, T \in 2^N$,

(P.1) if $i \notin S \cup T$ then $(S, x)\, I_i\, (T, y)$; and

(P.2) if $i \in S \cup T$ and $(S, x) \neq (T, y)$ then either $(S, x)\, P_i\, (T, y)$ or $(T, y)\, P_i\, (S, x)$.

Let $\mathcal{R}_i$ be the set of preferences of agent $i \in N$ over A satisfying (P.1) and (P.2), and let $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$ be the set of (preference) *profiles*. To emphasize agent i's preference or the preferences of agents in $S \subset N$ at profile R we often write it as (R_i, R_{-i}) or as (R_S, R_{-S}).

We denote by $[\varnothing]_i = \{(S, y) \in A \mid (S, y)\, I_i\, (\varnothing, x) \text{ for some } x \in X\}$ the subset of alternatives that agent i is indifferent to any alternative for which i is not a member of. By (P.1), $[\varnothing]_i$ is the indifference class generated by the empty society. With an abuse of notation we often treat, when listing a preference ordering, the indifference class $[\varnothing]_i$ as if it were an alternative; for instance, given R_i and (S, x) we write $(S, x)R_i[\varnothing]_i$ to represent that $(S, x)R_i(T, y)$ for all $(T, y) \in [\varnothing]_i$. Given a profile $R = (R_i)_{i \in N} \in \mathcal{R}$ and a subset of agents $S \subset N$ we denote by $R_{|S}$ the restriction of R to 2^S. Namely, given $i \in T \cap T'$, $T \cup T' \subset S$ and $x, y \in X$, $(T, x) \left(R_{|S}\right)_i (T', y)$ if and only if $(T, x)\, R_i\, (T', y)$.

Given a subset of alternatives $A' \subseteq A$ and a preference R_i, the *choice* of agent i in A' at R_i is the family of best subsets of A' according to R_i; namely,

$$C(A', R_i) = \{(S, x) \in A' \mid (S, x)\, R_i\, (T, y) \text{ for all } (T, y) \in A'\} .$$

We define three different sets that we will use in the sequel, all related to a preference R_i of agent i. The *top* of R_i, denoted by $\tau(R_i)$, is the set of all best alternatives according to R_i; namely,

$$\tau(R_i) = \{(S,x) \in A \mid (S,x)\, R_i\, (T,y) \text{ for all } (T,y) \in A\}.$$

Of course, $C(A, R_i) = \tau(R_i)$. The *lower counter set of* R_i *at* (S,x), denoted by $L((S,x), R_i)$, is the set of alternatives that are at least as bad as alternative (S,x) according to R_i; namely,

$$L((S,x), R_i) = \{(T,y) \in A \mid (S,x)\, R_i\, (T,y)\}.$$

The *upper counter set of* R_i *at* (S,x), denoted by $U((S,x), R_i)$, is the set of alternatives that are at least as good as alternative (S,x) according to R_i; namely,

$$U((S,x), R_i) = \{(T,y) \in A \mid (T,y)\, R_i\, (S,x)\}.$$

A *rule* is a social choice function $f : \mathcal{R} \to A$ selecting, for each profile $R \in \mathcal{R}$, an alternative $f(R) \in A$. To be explicit about the two components of the alternative chosen by f at R, we will often write $f(R)$ as $(f_N(R), f_X(R))$, where $f_N(R) \in 2^N$ and $f_X(R) \in X$.

To define anonymous and neutral rules let $\pi : N \to N$ and $\sigma : X \to X$ be permutations (one-to-one mappings) of the set of agents and the set of outcomes, respectively. Given $i \in N$ and $x \in X$, $\pi(i)$ is the agent assigned to i after applying the permutation π to N, and $\sigma(x)$ is the outcome assigned to x after applying the permutation σ to X. The set of all permutations $\pi : N \to N$ will be denoted by Π and the set of all permutations $\sigma : X \to X$ will be denoted by Σ. For $\pi \in \Pi$ and $1 \leq k \leq n$, we write π_k to denote the agent $\pi^{-1}(k)$. For $\sigma \in \Sigma$ and $x \in X$, we write σ_x to denote the outcome $\sigma^{-1}(x)$. Let $S \in 2^N$ be a subset of agents and π be a permutation of N. Denote by $\pi(S)$ the subset of agents associated to S by π; namely, $\pi(S) = \{i \in N \mid \pi(j) = i \text{ for some } j \in S\}$. Let $Y \subseteq X$ be a subset of outcomes and σ be a permutation of X. Denote by $\sigma(Y)$ the subset of outcomes associated to Y by σ; namely, $\sigma(Y) = \{x \in X \mid \sigma(y) = x \text{ for some } y \in Y\}$. Let $R \in \mathcal{R}$ be a profile and $\pi \in \Pi$ be a permutation of the set of agents N. Denote by R^π the new profile where for all $i \in N$, agent $\pi(i)$ has the preference R_i after replacing in the ordering R_i each pair (S,x) by $(\pi(S), x)$. Similarly, let $R \in \mathcal{R}$ be a profile and $\sigma \in \Sigma$ be a permutation of the set of outcomes X. Denote by R^σ the new profile where for all $i \in N$ the preference R_i^σ is obtained from R_i after replacing each pair (S,x) by $(S, \sigma(x))$.

Finally, to define a consistent rule we will have to specify how a given rule can be applied to a subprofile. One way of doing so it is to see a rule $f : \mathcal{R} \to A$ as it were a family of rules.

Given a nonempty subset $S \in 2^N \backslash \{\varnothing\}$, denote by $\mathcal{R}^S$ the set of subprofiles $R_{|S} = (R_{|i})_{i \in S}$ where each $R_{|i}$, $i \in S$, is defined over pairs in $2^S \times X$ and it is obtained by restricting R_i only to alternatives in $2^S \times X$. Thus, a rule f can be identified with the collection $\{f^S\}_{S \in 2^N \backslash \{\varnothing\}}$ of rules where for each $S \in 2^N \backslash \{\varnothing\}$, $f^S : \mathcal{R}^S \to 2^S \times X$. If no confusion can arise, we often omit the superscript S and write $f(R_{|S})$.

3 Properties of rules

In this section we present several properties that a rule may satisfy. The first three impose conditions on f at each profile.

A rule $f : \mathcal{R} \to A$ is efficient if it always selects a Pareto optimal allocation.

Efficiency For each $R \in \mathcal{R}$ there is no $(S, x) \in A$ with the property that $(S, x) R_i f(R)$ for all $i \in N$ and $(S, x) P_j f(R)$ for some $j \in N$.

A rule $f : \mathcal{R} \to A$ is unanimous if it selects an alternative in the intersection of all tops, whenever this intersection is nonempty.

Unanimity For all $R \in \mathcal{R}$ such that $\bigcap\limits_{i \in N} \tau(R_i) \neq \varnothing$, $f(R) \in \bigcap\limits_{i \in N} \tau(R_i)$.

The next property is related to the stability of a rule $f : \mathcal{R} \to A$, and captures the idea that agents are able to getting out of the society at their free will. Internal stability says that no agent belonging to the final society would prefer to leave it.

Internal stability For all $R \in \mathcal{R}$ and all $i \in f_N(R)$, $f(R) P_i [\varnothing]_i$.

The next seven properties impose conditions on a rule by comparing the alternatives chosen by the rule at two different profiles. A rule is strategy-proof if it is always in the best interest of agents to reveal their preferences truthfully; namely, truth-telling is a weakly dominant strategy in the direct revelation game induced by the rule.

Strategy-proofness For all $R \in \mathcal{R}$, all $i \in N$ and all $R'_i \in \mathcal{R}_i$, $f(R_i, R_{-i}) R_i f(R'_i, R_{-i})$.[1]

A rule is monotonic if the chosen alternative at a profile improves in the ordering of the preference of an agent, then the rule selects the same alternative in the new profile.

Monotonicity For all R and all $R'_i \in \mathcal{R}_i$ such that $L(f(R), R_i) \subset L(f(R), R'_i)$, $f(R) = f(R'_i, R_{-i})$.

[1] If otherwise, *i.e.* $f(R'_i, R_{-i}) P_i f(R_i, R_{-i})$, we will say that i manipulates f at profile (R_i, R_{-i}) via R'_i.

Since the set of indifferent alternatives for an agent coincides in all preferences, monotonicity could be reformulated in an equivalent way by stating that for all $R \in \mathcal{R}$ and all $R'_i \in \mathcal{R}_i$ such that $U(f(R), R_i) \supset U(f(R), R'_i)$, $f(R'_i, R_{-i}) = f(R)$.

A rule is nonbossy if an agent that is not a member of the chosen society at a profile changes his preferences and remains a nonmember then, the rule chooses the same alternative at the two profiles.

Nonbossiness For all $R \in \mathcal{R}$, all $i \in N$ and all $R'_i \in \mathcal{R}_i$ such that $i \notin f_N(R) \cup f_N(R'_i, R_{-i})$, $f(R'_i, R_{-i}) = f(R)$.

A rule is anonymous if is invariant with respect to permutations of the agents; *i.e.*, the names of the agents are not relevant to select the alternative.

Anonymity For all $R \in \mathcal{R}$ and all permutation $\pi \in \Pi$ of the set of agents, $f_N(R^\pi) = \pi(f_N(R))$ and $f_X(R^\pi) = f_X(R)$.

In our setting anonymity and efficiency are incompatible (no rule satisfies both properties). To see that consider the case where $N = \{1, 2\}$, $X = \{x\}$, and R_1 and R_2 are as follows: $(\{1\}, x) P_1 [\varnothing]_1 P_1 (N, x)$ and $(\{2\}, x) P_2 [\varnothing]_2 P_2 (N, x)$. If f is efficient, $f(R) \in \{(\{1\}, x), (\{2\}, x)\}$. Suppose $f(R) = (\{1\}, x)$; *i.e.*, $f_N(R) = \{1\}$ (the other case proceeds similarly, and hence we omit it). Consider the permutation π where $\pi(1) = 2$ and $\pi(2) = 1$. Since $\pi(\{1\}) = \{2\}$, $R^\pi = (R_2, R_1)$ and the sets of efficient alternatives at R and at R^π coincide, $f_N(R^\pi) = \{1\} \neq \{2\} = \pi(\{1\}) = \pi(f_N(R))$. Hence, f is not anonymous.

A rule is consistent if the following requirement holds. Apply the rule to a given profile and consider the subset of agents that are members of the chosen society. Construct the new subprofile of preferences restricted to this new set of chosen agents. Then, the rule does not require to modify the chosen alternative because when applied to the new subprofile the new alternative coincides with the alternative chosen at the original profile.

Consistency For all $R \in \mathcal{R}$, $f(R) = f\left(R_{|f_N(R)}\right)$.

A rule is neutral if the name of the outcomes do not play any role in selecting the social alternative.

Neutrality For all $R \in \mathcal{R}$ and all permutation $\sigma \in \Sigma$ of X, $f(R^\sigma) = (f_N(R), \sigma(f_X(R)))$.

A rule satisfies the property of participation if all agents prefer to be involved in the election of the social alternative rather than to exclude themselves by not submitting their preferences; namely, participation guarantees that the procedure to choose the social alternative is individually rational.

PARTICIPATION For all $R \in \mathcal{R}$ and all $i \in N$, $f(R)\, R_i [\varnothing]_i$.

4 Strategy-proof rules

In this section we want to characterize the class of all strategy-proof, unanimous and non-bossy rules.[2] They will be adaptations of the serial dictator rules to our setting. A serial dictator rule induced by $\pi \in \Pi$ and $x \in X$, denoted by $f^{\pi,x}$, proceeds as follows. Fix a profile $R \in \mathcal{R}$ and look for the best alternative (S_1, x_1) of agent π_1, the first in the ordering induced by π. If $\pi_1 \in S_1$, set $f^{\pi,x}(R) = (S_1, x_1)$. Otherwise, look for the best alternative (S_2, x_2) of agent π_2, the second in the ordering induced by π, with the property that $\pi_1 \notin S_2$. If $\pi_2 \in S_2$, set $f^{\pi,x}(R) = (S_2, x_2)$. Otherwise, look for the best alternative (S_3, x_3) of agent π_3, the third in the ordering induced by π, provided that $\pi_1, \pi_2 \notin S_3$, and so on. At the end, look for the best alternative (S_n, x_n) of agent π_n, the last in the ordering induced by π, with the property that for each $i \in \{1, ..., n-1\}$, $\pi_i \notin S_n$. If $\pi_n \in S_n$, set $f^{\pi,x}(R) = (S_n, x_n)$. Otherwise, and since no agent wants to stay in the society whatever element of X is selected, set $f^{\pi,x}(R) = (\varnothing, x)$. So, x plays the role of the residual outcome only when no agent wants to stay in the society under any circumstance.

We now define serial sequential rule formally. Fix $\pi \in \Pi$ and $x \in X$. Let $R \in \mathcal{R}$ be a profile. Define $f^{\pi,x}(R)$ recursively, as follows.

Stage 1. Let $A_1 = A$. Consider two cases:

1. $|C(A_1, R_{\pi_1})| = 1$. Then, $C(A_1, R_{\pi_1}) = \tau(R_{\pi_1})$. Set $(S_1, x_1) = C(A_1, R_{\pi_1})$ and observe that $\pi_1 \in S_1$. Define
$$f^{\pi,x}(R) = (S_1, x_1).$$

2. $|C(A_1, R_{\pi_1})| > 1$. Then, $C(A_1, R_{\pi_1}) = \{(S, x') \in A \mid \pi_1 \notin S \text{ and } x' \in X\}$. Go to Stage 2.

We now define Stage k $(1 < k < n)$, assuming that the stage $k-1$ has been reached and A_{k-1} was defined on it.

[2] Observe that the preferences we are considering satisfy (P.1) and hence, rules do not operate on the universal domain of preferences over $2^N \times X$. Thus, the Gibbard-Satterthwaite Theorem can not be applied (see Gibbard (1973) and Satterthwaite (1975)).

Stage k. Let $A_k = C(A_{k-1}, R_{\pi_{k-1}})$. Consider two cases.

1. $|C(A_k, R_{\pi_k})| = 1$. Then, $C(A_k, R_{\pi_k}) = \tau(R_{\pi_k})$. Set $(S_k, x_k) = C(A_k, R_{\pi_k})$ and observe that $\pi_k \in S_k$. Define

$$f^{\pi,x}(R) = (S_k, x_k).$$

2. $|C(A_k, R_{\pi_k})| > 1$. Then, $C(A_k, R_{\pi_k}) = \{(S, x') \in A \mid \pi_i \notin S \text{ for all } i \leq k \text{ and } x' \in X\}$. Go to Stage $k+1$.

We now define Stage n, the last stage of the procedure, assuming that the stage $n-1$ has been reached and A_{n-1} was defined on it.

Stage n. Let $A_n = C(A_{n-1}, R_{\pi_{n-1}})$. Consider two cases.

1. $|C(A_n, R_{\pi_n})| = 1$. Then, $C(A_n, R_{\pi_n}) = \tau(R_{\pi_n})$. Set $(S_n, x_n) = C(A_n, R_{\pi_n})$ and observe that $\pi_n \in S_n$. Define

$$f^{\pi,x}(R) = (S_n, x_n).$$

2. $|C(A_n, R_{\pi_n})| > 1$. Then, $C(A_n, R_{\pi_n}) = \{(\varnothing, x') \in A \mid x' \in X\}$. Define

$$f^{\pi,x}(R) = (\varnothing, x).$$

Example 1 below illustrates this procedure.

Example 1 Let $N = \{1, 2\}$ be the set of agents, $X = \{a, b, c\}$ be the set of outcomes and consider the identity permutation $\pi = (\pi_1, \pi_2) = (1, 2)$ and $x = a$. We apply the serial dictator rule $f^{(1,2),a}$ to the following preferences, where we give the list of the alternatives in decreasing order from the top and we only order the alternatives needed to compute $f^{(1,2),a}$ at some profiles.

R_1	R'_1	R_2	R'_2
(N,b)	$\{(S,y) \in A \mid 1 \notin S, y \in A\}$	(N,a)	(N,a)
		(N,b)	(N,b)
		$(\{2\},c)$	$\{(S,y) \in A \mid 2 \notin S, y \in A\}$

.

Then,

$$\begin{aligned} f^{(1,2),a}(R_1,R_2) &= (N,b), \\ f^{(1,2),a}(R_1,R_2') &= (N,b) \\ f^{(1,2),a}(R_1',R_2) &= (\{2\},c), \text{ and} \\ f^{(1,2),a}(R_1',R_2') &= (\varnothing,a). \end{aligned}$$

■

We are now ready to state as Theorem 1 the characterization of the class of all strategy-proof, unanimous and nonbossy rules as the family of all serial dictator rules. The Appendix at the end of the paper contains the proof of Theorem 1 and three examples of rules indicating the independence of the three properties used in the characterization.

Theorem 1 *Assume $|X| \geq 3$. A rule $f : \mathcal{R} \to A$ is strategy-proof, unanimous and non-bossy if and only if f is a serial dictator rule for some permutation $\pi \in \Pi$ and alternative $x \in X$.*

5 Consistent and internally stable rules

By Theorem 1, if we insist on strategy-proofness together with the two additional weak requirements of unanimity and nonbossiness, we have to use a serial dictatorial rule. In this section, we consider situations where the strategic manipulation in the preference revelation game is not an issue and will look for rules satisfying two meaningful properties in our setting, assuming agents report truthfully their preferences. Internal stability (no agent, member of the chosen society, wants to leave it) is an specially interesting property because in most societies, agents are not obliged to stay in the society if they want to leave it. The second property is consistency. Assume that the rule f has selected the pair (S,x) at $R \in \mathcal{R}$. Thus, agents in S might want to reconsider again the alternative (S,x). Consistency says that if f is applied to $R_{|S}$, the pair (S,x) would be also selected. Hence, members of the new society S do not need to reconsider the choice (S,x) of the former society N.

To look for consistent rules satisfying also internal stability we ask whether three of the most prominent rules in classical social choice satisfy them. Recall that in the classical setting the goal is to select an outcome, from a given set X, taking into account the strict preferences of agents over X. The rules we consider are:

1. Approval voting. Each agent $i \in N$ votes for a subset X_i of X. For each outcome $x \in X$, compute the number of received votes; namely, $|\{i \in N : x \in X_i\}|$. The outcome with more votes is selected. A tie-breaking rule should be applied whenever several outcomes obtain the largest number of votes.

2. Plurality voting. Each agent $i \in N$ votes for an outcome $x_i \in X$. The outcome with more votes is selected. A tie-breaking rule should be applied whenever several outcomes obtain the largest number of votes.

3. Borda method. Each agent $i \in N$ ranks all outcomes. Assign a preestablished number of points to each outcome depending on its position in the order. For each outcome, compute the sum, over all agents, of the points obtained by such outcome. Select the outcome with more points. A tie-breaking rule should be applied whenever several outcomes obtain the largest number of points.

We adapt the three voting methods to our setting, where the set of alternatives is $2^N \times X$. In addition, we will have to deal with the indifferences arising from property (P.1) of preference relations.

1. Approval voting. Each agent $i \in N$ votes for all pairs (S, x) such that $(S, x)\, P_i\, [\varnothing]_i$.

2. Plurality voting. Each agent $i \in N$ votes for his top alternative, $\tau(R_i)$. If $\tau(R_i) = [\varnothing]_i$, assume that i votes for all pairs $(S, x) \in [\varnothing]_i$.

3. Borda method. For each agent $i \in N$, consider $[\varnothing]_i$ as a single alternative in i' rank. For each pair $(S, x) \in A$ and each $i \in N \backslash S$, assign to the pair (S, x) the score obtained by $[\varnothing]_i$.

Example 2 below shows that none of these extensions satisfy internal stability.

Example 2 Let $R \in \mathcal{R}$ and $x \in X$ be such that for all $i \in N \backslash \{1\}$, $\tau(R_i) = (N, x)$, $\tau(R_1) = [\varnothing]_1$ and for each (S, y) with $1 \in S$, $(N, x)\, R_1\, (S, y)$. Then, the three adapted voting methods choose (N, x) at R. Nevertheless, (N, x) is not internal stable because agent 1 prefers to leave the society. ■

Since we are interested in identifying rules satisfying internal stability, we modify the previous methods by considering only pairs (S, x) that are internally stable for each $i \in S$ according to R_i; namely, $(S, x)\, P_i\, [\varnothing]_i$ for each $i \in S$. In approval voting agents vote only for pairs that are internally stable. In plurality voting each agent votes for his best internally stable pair. In a Borda method we consider only the rank, given by the preference, among the internally stable pairs. With these modifications the three methods satisfy internal stability by definition. Denote by f^{AV}, f^P and f^B Approval voting, the Plurality voting, and the Borda method, respectively.

Our first result is negative: plurality voting and Borda method do not satisfy consistency. To see that, consider Example 3 below.

Example 3 Let $N = \{1, 2, 3, 4, 5, 6\}$ and $X = \{y_1, y_2, y_3, y_4, y_5\}$ and consider the following profile $R \in \mathcal{R}$. For each $i \in N$, $(S, x)\ P_i\ [\varnothing]_i$ whenever $i \in S$ (namely, all pairs are internally stable). In addition, R is one among all those profiles satisfying the following properties, where the first column indicates the rank of each of the six preference relations.

	R_1	R_2	R_3	R_4	R_5	R_6
First	(N, y_1)	(N, y_2)	(N, y_3)	$(N\backslash\{6\}, y_4)$	$(N\backslash\{6\}, y_4)$	(N, y_5)
Second	$(N\backslash\{6\}, y_1)$	$(N\backslash\{6\}, y_1)$	$(N\backslash\{6\}, y_1)$	$(N\backslash\{1\}, y_4)$	$(N\backslash\{1\}, y_4)$	
Third	$(N\backslash\{6\}, y_4)$	$(N\backslash\{6\}, y_4)$	$(N\backslash\{6\}, y_4)$	$(N\backslash\{2\}, y_4)$	$(N\backslash\{2\}, y_4)$	
Fourth				$(N\backslash\{3\}, y_4)$	$(N\backslash\{3\}, y_4)$	
Fifth				$(N\backslash\{6\}, y_1)$	$(N\backslash\{6\}, y_1)$	

First, plurality voting does not satisfy consistency since $f^P(R) = (N\backslash\{6\}, y_4)$ but at the same time $f^P\left(R_{|N\backslash\{6\}}\right) = (N\backslash\{6\}, y_1)$. Consider now the classical definition of the Borda method where the scores from the worst to the best alternative are given by $0, 1, 2, ..., k-2, k-1$, where k is the number of available alternatives. It is possible to select a profile R' satisfying the above rankings in such a way that $f^B(R') = (N\backslash\{6\}, y_4)$. Besides, $f^B\left(R'_{|N\backslash\{6\}}\right) = (N\backslash\{6\}, y_1)$. Hence, this Borda method does not satisfy consistency. ■

Fortunately, approval voting satisfies not only consistency but also other desirable properties. Before stating this result formally we propose a tie-breaking rule, to be used whenever more than one alternative obtains the highest number of votes. Let ρ be a monotonic order over the family of subsets of 2^N. Namely, given $S, T \in 2^N$ such that $S \subset T$, $T\rho S$. Observe that $N\rho S$ for all $S \neq N$.

Fix a monotonic order ρ over 2^N. Denote by $f^{AV,\rho}$ the approval voting that uses ρ to break the ties. Formally, let $A' = \{(S_k, x_k)\}_{k=1}^K$ be the set of alternatives that have received the largest number of votes according to approval voting at profile R. First select the society $S \in \{S_1, ..., S_K\}$ ranked higher by ρ and consider the subset of alternatives $\{(S_{k'}, x_{k'}) \in A' \mid S_{k'} = S\}$. Select now agent $i \in S$ ranked higher by ρ (as a singleton set) and choose finally at R the alternative that is most preferred by i among those in the family $\{(S_{k'}, x_{k'}) \in A' \mid S_{k'} = S\}$.

Proposition 1 below states that any Approval voting $f^{AV,\rho}$ satisfies internal stability and consistency, together with other desirable properties.

Proposition 1 *Let ρ be a monotonic order over 2^N. Then, the Approval voting $f^{AV,\rho}$ satisfies internal stability, consistency, efficiency, neutrality and participation. Moreover, in the subdomain of profiles where no tie-breaking rule is needed, $f^{A,\rho}$ also satisfies anonymity.*

Proof of Proposition 1 Observe that if (S, x) is approved by agent $i \in N$, then $i \in S$. This fact will be repeatedly used in the proof to show that $f^{AV,\rho}$ satisfies the properties, which we consider separately.

- Internal stability. By definition, $f^{A,\rho}$ satisfies internal stability.

- Consistency. Let $R \in \mathcal{R}$ be an arbitrary profile and let $(S, x) \in A$ be such that $S \subset f_N^{AV,\rho}(R)$. The set of agents approving (S, x) at R coincides with the set of agents approving (S, x) at $R_{|f_N^{A,\rho}(R)}$. Hence, it follows that $f^{AV,\rho}(R_{|f_N^{A,\rho}(R)}) = f^{AV,\rho}(R)$. Thus, $f^{AV,\rho}$ satisfies consistency.

- Efficiency. Suppose otherwise; namely, there exist $R \in \mathcal{R}$ and $(S, x) \in A$ such that $(S, x) R_i f^{AV,\rho}(R)$ for all $i \in N$ and $(S, x) \neq f^{AV,\rho}(R)$. Let $i \in f_N^{AV,\rho}(R)$. Since $f^{AV,\rho}$ satisfies internal stability, $f^{AV,\rho}(R) P_i [\varnothing]_i$. Hence, $i \in S$ and $(S, x) P_i f^{AV,\rho}(R)$. We consider two cases. First, assume $f_N^{AV,\rho}(R) \subsetneq S$. Since for each $i \in S \backslash f_N^{AV,\rho}(R)$, $f^{AV,\rho}(R) = [\varnothing]_i$ and $(S, x) R_i f^{AV,\rho}(R)$ it follows that $(S, x) P_i [\varnothing]_i$. Thus, all agents in S approve (S, x), which contradicts the definition of $f^{AV,\rho}(R)$. Second, assume $f_N^{AV,\rho}(R) = S$. Thus, $f^{A,V\rho}(R) = (S, y)$ with $y \neq x$ and all agents in S approve both, (S, x) and (S, y). Hence, the tie-breaking rule ρ has been used to select $f^{AV,\rho}(R)$. Thus, there exists $i \in S$ such that $f^{AV,\rho}(R) P_i (S, x)$ which is a contradiction.

- Neutrality. Let $R \in \mathcal{R}$ be a profile and σ a permutation of X. Observe that for any alternative (S, x) the number of agents approving (S, x) at R coincides with the number of agents approving (S, σ_x) at R^σ. We consider two cases. First, assume it is not necessary to apply ρ to select $f^{AV,\rho}(R)$. Namely, $f^{AV,\rho}(R)$ has been approved at R by more agents that any other alternative (S, x). Thus, $(f_N^{AV,\rho}(R), \sigma_{f_X^{AV,\rho}(R)})$

has been approved at R^{σ} by more agents that any other alternative (S, x). Hence, $f^{AV,\rho}(R^{\sigma}) = (f_N^{AV,\rho}(R), \sigma_{f_X^{AV,\rho}(R)})$. Second, assume it is necessary to apply ρ to select $f^{AV,\rho}(R)$. Let $\{(S_k, x_k)\}_{k=1}^{K}$ be the set of alternatives receiving the largest number of votes at R. Thus, $\{(S_k, \sigma_{x_k})\}_{k=1}^{K}$ is the set of alternatives receiving the largest number of votes at R^{σ}. Hence, $f_N^{AV,\rho}(R) = f_N^{AV,\rho}(R^{\sigma})$. Now, let $i \in f_N^{AV,\rho}(R)$ be the agent with the highest ranking according to ρ (as a singleton set) and let $i' \in f_N^{AV,\rho}(R^{\sigma})$ be the agent with the highest ranking according to ρ. Obviously, $i' = i$. Thus, $f_X^{AV,\rho}(R^{\sigma}) = \sigma_{f_X^{AV,\rho}(R)}$.

- Participation. We prove that $f^{AV,\rho}$ satisfies participation by showing that any rule satisfying internal stability also satisfies participation. Let $R \in \mathcal{R}$ be a profile and $i \in N$. Since $f_{N\backslash\{i\}}^{AV,\rho}\left(R_{|N\backslash\{i\}}\right) \subset N\backslash\{i\}$, $f_{N\backslash\{i\}}^{AV,\rho}\left(R_{|N\backslash\{i\}}\right) = [\varnothing]_i$. We distinguish between two cases. First, $i \in f_N^{AV,\rho}(R)$. Since $f^{AV,\rho}$ satisfies internal stability, $f^{AV,\rho}(R) \, P_i \, [\varnothing]_i$. Second, $i \notin f_N^{AV,\rho}(R)$. Thus, $f^{AV,\rho}(R) = [\varnothing]_i$.

- Anonymity in the profiles where no tie-breaking rule is needed. Assume that to select the alternative at profile R the tie-breaking ρ is not used. Then, $f^{AV,\rho}(R)$ has been approved at R by more agents than any other alternative (S, x). Hence, the number of agents approving (S, x) at R coincides with the number of agents approving $(\pi(S), x)$ at R^{π}. Thus, $(\pi(f_N^{AV,\rho}(R)), f_X^{AV,\rho}(R))$ has been approved at R^{π} by more agents that any other pair (S, x). Hence, $f^{AV,\rho}(R^{\pi}) = \left(\pi(f^{AV,\rho}(R)), f_X^{AV,\rho}(R)\right)$, which means that $f^{AV,\rho}$ satisfies anonymity at profile R. ■

We end this section by applying the Condorcet winner to our setting. First, we recall the definition of the Condorcet winner at a profile (over the set of outcomes) in the classical setting. Fix a profile over X and $x, y \in X$. We say that x beats y if the number of agents preferring x to y is larger that the number of agents preferring y to x. We say that x is a Condorcet winner (at a profile over X) if there is no y such that y beats x. It could be the case that no Condorcet winners exists or that there are several Condorcet winners (at a profile over X). Thus, the Condorcet winner is not a rule according to our definition.

We adapt the notion of a Condorcet winner to our setting as we have already did for the previous three rules. Fix a profile $R \in \mathcal{R}$ and two different alternatives (S, x) and (T, y). All agents in the set $S \cup T$ strictly prefer one alternative to the other one while all agents in the set $N\backslash(S \cup T)$ are indifferent between (S, x) and (T, y). Thus, (S, x) beats (T, y) at R

if the number of agents strictly preferring (S, x) to (T, y) is larger than the number of agents strictly preferring (T, y) to (S, x). In order to ensure that the chosen alternative satisfies internal stability at R we only consider alternatives (S, x) satisfying internal stability at R (namely, for all $i \in S$, $(S, x) P_i [\varnothing]_i$). When several Condorcet winners exist we apply the same tie-breaking rule ρ as in approval voting.

We say that a profile $R \in \mathcal{R}$ is *resolute* if there is an alternative $(S, x) \in A$ such that (S, x) beats (T, y) for all $(T, y) \neq (S, x)$. Thus, the Condorcet winner selects (S, x) at R. Let $f^{C,\rho}(R)$ denote the Condorcet winner (if any) at R. If profile R is resolute then, $f^{C,\rho}(R)$ is independent of ρ and $\left|f^{C,\rho}(R)\right| = 1$. Proposition 2 states that the Condorcet winner at resolute profiles satisfies the same properties as approval voting, at such profiles.

Proposition 2 *Let R be a resolute profile. Then, $f^{C,\rho}(R)$ satisfies internal stability, consistency, efficiency, anonymity, neutrality, and participation at R.*

Proof of Proposition 2 Fix a resolute profile R and set $f^{C,\rho}(R) = (S, x)$. We show that $f^{C,\rho}(R)$ satisfies the properties at R.

- Internal stability. By definition, $f^{C,\rho}(R)$ satisfies internal stability at R.

- Consistency. We prove that $f^{C,\rho}\left(R_{|S}\right) = (S, x)$ by showing that at $R_{|S}$, (S, x) beats (T, y) for all $(T, y) \neq (S, x)$ with $T \subset S$. Let (T, y) be an alternative with the above properties. Since (S, x) beats (T, y) at R, the number of agents in N preferring (S, x) to (T, y) is larger than the number of agents in N preferring (T, y) to (S, x). Moreover, each agent in $N \backslash S$ is indifferent between (S, x) and (T, y). Thus the number of agents in S preferring (S, x) to (T, y) (or (T, y) to (S, x)) coincides with the number of agents in N preferring (S, x) to (T, y) (or (T, y) to (S, x)). Hence, (S, x) beats (T, y) at $R_{|S}$.

- Efficiency. Suppose otherwise; namely, there exists (T, y) such that $(T, y) R_i (S, x)$ for all $i \in N$ and $(S, x) \neq (T, y)$. Let $i \in S$. Since (S, x) satisfies internal stability, $(S, x) P_i [\varnothing]_i$. Hence, $i \in T$ and $(T, y) P_i (S, x)$. Each agent in $N \backslash T$ is indifferent between (S, x) and (T, y). Thus (T, y) beats (S, x), which is a contradiction.

- Anonymity. Observe that $(\pi(S), x)$ beats $(\pi(T), y)$ at R^π, for each $(T, y) \neq (S, x)$. Hence, $f^{C,\rho}(R^\pi) = (\pi(S), x)$, which means that $f^{C,\rho}$ satisfies anonymity at profile R.

- Neutrality. Observe that (S, σ_x) beats (T, σ_y), at R^σ, for each $(T, y) \neq (S, x)$. Hence, $f^{C,\rho}(R^\sigma) = (S, \sigma_x)$, which means that $f^{C,\rho}$ satisfies neutrality at R.

- Participation. It holds because we already proved in the proof of Proposition 2 that if a rule satisfies internal stability, then it also satisfies participation. ■

Nevertheless, for profiles R that are not resolute the Condorcet winner $f^{C,\rho}(R)$, even when it is unique, may not satisfy consistency. To see that, consider the following example.

Example 4 Let $N = \{1,2,3,4,5\}$, $X = \{y_1, y_2\}$, and let ρ be any monotonic order satisfying $\{1\}\,\rho\,\{2\}\,\rho\,\{3\}\,\rho\,\{4\}$. In addition, take any profile R from all those satisfying the following properties, where the first column indicates the rank of each of the five preference relations.

	R_1	R_2	R_3	R_4	R_5
First	$(N\backslash\{5\}, y_1)$	$(N\backslash\{5\}, y_1)$	$(N\backslash\{5\}, y_2)$	$(N\backslash\{5\}, y_2)$	(N, y_1)
Second	$(N\backslash\{5\}, y_2)$	$(N\backslash\{5\}, y_2)$	(N, y_1)	(N, y_1)	$[\varnothing]_5$
Third	(N, y_1)	(N, y_1)	$(N\backslash\{5\}, y_1)$	$(N\backslash\{5\}, y_1)$	
Fourth	$[\varnothing]_1$	$[\varnothing]_2$	$[\varnothing]_3$	$[\varnothing]_4$	

The only internally stable alternatives are $(N\backslash\{5\}, y_1)$, $(N\backslash\{5\}, y_2)$, and (N, y_1). Notice that, at R, $(N\backslash\{5\}, y_1)$ is tied with $(N\backslash\{5\}, y_2)$, $(N\backslash\{5\}, y_2)$ beats (N, y_1) and (N, y_1) beats $(N\backslash\{5\}, y_1)$. Since there exists a unique Condorcet winner, $(N\backslash\{5\}, y_2)$, it must be the case that $f^{C,\rho}(R) = (N\backslash\{5\}, y_2)$. The subprofile $R_{|N\backslash\{5\}}$ is given by

	$R_{\vert N\backslash\{5\}_1}$	$R_{\vert N\backslash\{5\}_2}$	$R_{\vert N\backslash\{5\}_3}$	$R_{\vert N\backslash\{5\}_4}$
First	$(N\backslash\{5\}, y_1)$	$(N\backslash\{5\}, y_1)$	$(N\backslash\{5\}, y_2)$	$(N\backslash\{5\}, y_2)$
Second	$(N\backslash\{5\}, y_2)$	$(N\backslash\{5\}, y_2)$	$(N\backslash\{5\}, y_1)$	$(N\backslash\{5\}, y_1)$
Third	$[\varnothing]_1$	$[\varnothing]_2$	$[\varnothing]_3$	$[\varnothing]_4$

At $R_{|N\backslash\{5\}}$, $(N\backslash\{5\}, y_1)$ is tied with $(N\backslash\{5\}, y_2)$. Thus, applying the tie-breaking rule ρ, and since agent 1 prefers $(N\backslash\{5\}, y_1)$ to $(N\backslash\{5\}, y_2)$, we have that $f^{C,\rho}(R) = (N\backslash\{5\}, y_1)$, which means that $f^{C,\rho}$ does not satisfy consistency. ■

References

[1] Barberà, S., M. Maschler and J. Shalev (2001): "Voting for Voters: A Model of Electoral Evolution," *Games and Economic Behavior* 37, 40–78.

[2] Barberà, S. and A. Perea (2002): "Supporting others and the evolution of influence," *Journal of Economic Dynamics & Control* 26, 2051–2092.

[3] Barberà, S., H. Sonnenschein and L. Zhou (1991): "Voting by committees," *Econometrica* 59, 595–609.

[4] Berga, D., G. Bergantiños, J. Massó and A. Neme (2004): "Stability and voting by committees with exit," *Social Choice and Welfare* 23, 229–247.

[5] Berga, D., G. Bergantiños, J. Massó and A. Neme (2006): "On exiting after voting," *International Journal of Game Theory* 34: 33–54.

[6] Bergantiños, G., J. Massó and A. Neme (2015): "The division problem under constraints," *Games and Economic Behavior* 89, 56–77.

[7] Gibbard, A. (1973): "Manipulation of voting schemes: a general result," *Econometrica* 41, 587–601.

[8] Satterthwaite, M. (1975): "Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions," *Journal of Economic Theory* 10, 187–217.

[9] Thomson, W. (1994): "Consistent solutions to the problem of fair division when preferences are single-peaked," *Journal of Economic Theory* 63, 219–245.

[10] Thomson, W. (2007): "On the existence of consistent rules to resolve conflicting claims: a geometric approach," *Review of Economic Design* 11, 225–251.

Appendix: The proof of Theorem 1

Theorem 1 *Assume $|X| \geq 3$. A rule $f : \mathcal{R} \to A$ is strategy-proof, unanimous and nonbossy if and only if f is a serial dictator rule for some permutation $\pi \in \Pi$ and alternative $x \in X$.*

We start by presenting an additional notion, that will be used in the sequel. Given a rule $f : \mathcal{R} \to A$, an agent $i \in N$ and a profile $R \in \mathcal{R}$ the *option set* of agent i at R, denoted by $o(i, R)$, is the set of alternatives that may be chosen by f when the other agents declare the subprofile R_{-i}; namely,

$$o(i, R) = \{(S, x) \in A \mid (S, x) = f(R'_i, R_{-i}) \text{ for some } R'_i \in \mathcal{R}_i\}.$$

Notice that the option set of agent i at R does not depend on the preference R_i. We use the full profile R just for notational convenience.

We proceed by presenting some lemmata that will be used in the proof of Theorem 1.

Lemma 1 *Let $f : \mathcal{R} \to A$ be a strategy-proof, unanimous and nonbossy rule. Then, the following hold.*

(1) *f satisfies monotonicity.*

(2) *f satisfies efficiency.*

(3) *For all $R \in \mathcal{R}$ and $i \in N$, $f(R) = C(o(i, R), R_i)$.*

Proof of Lemma 1 Assume that $f : \mathcal{R} \to A$ is strategy-proof, unanimous and nonbossy. We prove the three statements.

(1) Suppose $R \in \mathcal{R}$, $i \in N$, and $R'_i \in \mathcal{R}_i$ are such that $L(f(R), R_i) \subset L(f(R), R'_i)$ and $f(R) \neq f(R'_i, R_{-i})$. Three cases are possible:

1. $f(R) P_i f(R'_i, R_{-i})$. Since $L(f(R), R_i) \subset L(f(R), R'_i)$, $f(R'_i, R_{-i}) \in L(f(R), R'_i)$ and hence $f(R) P'_i f(R'_i, R_{-i})$. Thus, i manipulates f at profile (R'_i, R_{-i}) via R_i, which contradicts strategy-proofness of f.

2. $f(R'_i, R_{-i}) P_i f(R)$. This contradicts strategy-proofness of f since i manipulates f at profile R via R'_i.

3. $f(R'_i, R_{-i}) I_i f(R)$. Then, by (P.2), $i \notin f(R'_i, R_{-i}) \cup f(R)$. By nonbossiness of f, $f(R'_i, R_{-i}) = f(R)$ which is a contradiction.

(2) Suppose f is not efficient. Namely, there exist $R \in \mathcal{R}$ and $(T, y) \in A$ such that $(T, y) R_i f(R)$ for all $i \in N$ and $(T, y) P_j f(R)$ for some $j \in N$. Let $R' \in \mathcal{R}$ be the profile such that for each $i \in N$, $\tau(R_i') = \{(T', y') \in A \mid (T', y') I_i (T, y)\}$ and orders the rest of alternatives as R_i does. Consider the profile $(R_1', R_{-1}) \in \mathcal{R}$ and suppose that $f(R_1', R_{-1}) \neq f(R)$. If $f(R_1', R_{-1}) I_1 f(R)$ then $1 \notin f_1(R_1', R_{-1}) \cup f_1(R)$ but this contradicts nonbossiness of f. If $f(R_1', R_{-1}) P_1 f(R)$ then, f is not strategy-proof. If $f(R) P_1 f(R_1', R_{-1})$ then $f(R) P_1' f(R_1', R_{-1})$, which means that 1 manipulates f at (R_1', R_{-1}) via R_1. Repeating this argument sequentially for agents $i = 2, ..., n$ we obtain that $f(R') = f(R)$. But this contradicts unanimity of f because $(T, y) \in \bigcap_{i \in N} \tau(R_i)$.

(3) Let $R \in \mathcal{R}$ and $i \in N$ be arbitrary and consider $(S, x) \in C(o(i, R), R_i)$. Then, $(S, x) R_i f(R)$. Assume $f(R) \neq (S, x)$. Two cases are possible:

1. $i \in S$. Then, $(S, x) P_i f(R)$. Since $(S, x) \in o(i, R)$, $(S, x) = f(R_i', R_{-i})$ for some $R_i' \in \mathcal{R}_i$, which means that i manipulates f at profile R via R_i'. A contradiction.

2. $i \notin S$. By nonbossiness, $i \in f_N(R)$ and hence, $(S, x) P_i f(R)$. Now, we obtain a contradiction with strategy-proofness of f by proceeding in a similar way as we did in the previous case. ■

For the next step, it will be useful to consider the set $\mathcal{F}$ of all complete, transitive and antisymmetric binary relations over X. Namely, $\mathcal{F}$ can be seen as the set of all strict preferences over X. Now, for each strict preference $\succ$ over X, each $N^* \subset N$, and each agent $i \in N$ we associate a preference over $2^N \times X$ (namely, an element of $\mathcal{R}_i$) by means of a mapping $\varphi^{N^*,i} : \mathcal{F} \to \mathcal{R}_i$, where for each $\succ \in \mathcal{F}$ we select a particular preference $\varphi^{N^*,i}(\succ) \in \mathcal{R}_i$, denoted by $R_{\varphi^{N^*,i}(\succ)}$, among those satisfying the following features.

- If $i \in N^*$, consider several cases:
  - If $i \in S \cap T \subset N^*$ then $(S, x) P_{\varphi^{N^*,i}(\succ)} (T, y)$ if and only if $x \succ y$.
  - If $i \in T \subset S \subset N^*$ then $(S, x) P_{\varphi^{N^*,i}(\succ)} (T, x)$.
  - If $i \in S \subset N^*$ then $(S, x) P_{\varphi^{N^*,i}(\succ)} (\varnothing, y)$ for all $x, y \in X$.
  - If $i \in S$ and $S \cap (N \backslash N^*) \neq \varnothing$ then $(\varnothing, x) P_{\varphi^{N^*,i}(\succ)} (S, y)$ for all $x, y \in X$.
- If $i \notin N^*$, consider only the case $i \in S$. Then, $(\varnothing, x) P_{\varphi^{N^*,i}(\succ)} (S, y)$ for all $x, y \in X$.[3]

[3]Even $R_{\varphi^{N^*,i}(\succ)}$ could not depend on $\succ$, for simplicity we maintain the notation $R_{\varphi^{N^*,i}(\succ)}$.

We should note that there are many preferences in $\mathcal{R}_i$ satisfying the above conditions. We just select one of them, and call it $\varphi^{N^*,i}(\succ)$.

Fix $N^* \subseteq N$ and define a social choice function $g : \mathcal{F}^{N^*} \to X$ as follows. For each subprofile $(\succ_i)_{i\in N^*} \in \mathcal{F}^{N^*}$ of preferences over X set

$$g((\succ_i)_{i\in N^*}) = f_X((R_{\varphi^{N^*,i}(\succ_i)})_{i\in N}).$$

Lemma 2 *Let $f : \mathcal{R} \to A$ be a strategy-proof, unanimous and nonbossy rule. Then, the social choice function g is dictatorial; i.e., there exists $j \in N^*$ such that for all $(\succ_i)_{i\in N^*}$, $g((\succ_i)_{i\in N^*}) = \tau(\succ_j)$ where $\tau(\succ_j) \succ_j y$ for all $y \in X\backslash\{\tau(\succ_j)\}$.*

Proof of Lemma 2 Since g is defined on the universal domain of strict preference profiles on X, the Gibbard-Satterthwaite Theorem says that if g is onto (for each $x \in X$, there exists $(\succ_i)_{i\in N^*}$ such that $g((\succ_i)_{i\in N^*}) = x$) and strategy-proof then, g is dictatorial.

We first prove that g is onto. Let $x \in X$ and $(\succ_i)_{i\in N^*}$ be such that for each $i \in N^*$, $\tau(\succ_i) = x$. By definition of $R_{\varphi^{N^*,i}(\succ_i)}$, $\tau(R_{\varphi^{N^*,i}(\succ_i)}) = (N^*, x)$ if $i \in N^*$ and $(N^*, x) \in \tau\left(R_{\varphi^{N^*,i}(\succ_i)}\right)$ if $i \notin N^*$. Since f satisfies unanimity and $\bigcap_{i\in N} \tau\left(R_{\varphi^{N^*,i}(\succ_i)}\right) = (N^*, x)$, we have that $f\left(\left(R_{\varphi^{N^*,i}(\succ_i)}\right)_{i\in N}\right) = (N^*, x)$. Thus, $g\left((\succ_i)_{i\in N^*}\right) = f_X\left(\left(R_{\varphi^{N^*,i}(\succ_i)}\right)_{i\in N}\right) = x$.

We now prove that g is strategy-proof. Suppose otherwise. Then, there exist $(\succ_i)_{i\in N^*}$, $j \in N^*$ and $\succ'_j$ such that $g(\succ'_j, \succ_{-j}) \succ_j g(\succ_j, \succ_{-j})$. By definition of g, $f_X\left(\left(R_{\varphi^{N^*,i}(\succ_i)}\right)_{i\in N}\right) = g(\succ_j, \succ_{-j})$ and $f_X\left(R_{\varphi^{N^*,j}(\succ'_j)}, \left(R_{\varphi^{N^*,i}(\succ_i)}\right)_{i\neq j}\right) = g(\succ'_j, \succ_{-j})$. By definition of $R_{\varphi^{N^*,i}(\succ_i)}$ we know that for each $i \in N^*$, each $\succ \in \mathcal{F}$, each $x \in X$, and each $S \subset N$, $S \neq N^*$, we have that $(N^*, x)\, P_{\varphi^{N^*,i}(\succ)}\, (S, x)$. Besides, for each $i \in N\backslash N^*$, each $\succ \in \mathcal{F}$, each $x \in X$, and each $S \subset N$ with $i \in S$ we have that $(N^*, x)\, P_{\varphi^{N^*,i}(\succ)}\, (S, x)$. Since f is efficient,

$$\begin{aligned} f((R_{\varphi^{N^*,i}(\succ_i)})_{i\in N}) &= (N^*, g(\succ_j, \succ_{-j})) \text{ and} \\ f(R_{\varphi^{N^*,j}(\succ'_j)}, (R_{\varphi^{N^*,i}(\succ_i)})_{i\neq j}) &= \left(N^*, g(\succ'_j, \succ_{-j})\right). \end{aligned}$$

By definition of $\varphi^{N^*,j}(\succ_j)$,

$$\left(N^*, g(\succ'_j, \succ_{-j})\right) P_{\varphi^{N^*,j}(\succ_j)} \left(N^*, g\left(\succ_j, \succ_{-j}\right)\right),$$

which contradicts that f is strategy -proof. ■

Fix $R_i \in \mathcal{R}_i$ and $(S, x) \in A$. Denote by ${}_{(S,x)}R_i$ the preference over A obtained from R_i by placing (S, x) and all its indifferent alternatives (if any) at the bottom of the ordering. Formally, ${}_{(S,x)}R_i$ is defined so that $(T, y)\,{}_{(S,x)}R_i(S, x)$, for every $(T, y) \in A$ and for every (T,y), $(T',y') \in A \backslash \{(S,x)\}$, $(T,y)_{(S,x)} R_i (T',y')$ if and only if $(T,y) R_i (T',y')$. Similarly, ${}^{(S,x)}R_i$ denotes the preference over A obtained from R_i by placing (S,x) and all its indifferent alternatives (if any) at the top of the ordering. Formally, ${}^{(S,x)}R_i$ is defined so that $(S,x)^{(S,x)}R_i(T,y)$, for every $(T,y) \in A$ and for every (T,y), $(T',y') \in A \backslash \{(S,x)\}$, $(T,y)^{(S,x)}R_i(T',y')$ if and only if $(T,y)R_i(T',y')$.

Lemma 3 *Let $f : \mathcal{R} \to A$ be a strategy-proof, unanimous and nonbossy rule and let $R \in \mathcal{R}$ and $i, j \in S \subseteq N$ be such that $i \neq j$, $f(R) = (S,x)$ and $|o(i,R)| \geq 3$. Then $|o(j,R)| = 1$.*

Proof of Lemma 3 Suppose $|o(j,R)| \geq 2$ holds. This means that we can find $(T,y) \in o(i,R) \setminus \{(S,x)\}$ and $(U,z) \in o(j,R) \setminus \{(S,x)\}$ such that $(T,y) \neq (U,z)$. Consider any preference $R_i' \in \mathcal{R}$, where

$$R_i' = \begin{cases} {}_{(U,z)}R_i & \text{if } (S,x)P_i(U,z) \\ {}^{(U,z)}R_i & \text{if } (U,z)P_i(S,x). \end{cases}$$

Notice that $(U,z)I_i(S,x)$ does not hold since $i \in S$. Besides, define $R_j' \in \mathcal{R}$, where

$$R_j' = \begin{cases} {}_{(T,y)}R_j & \text{if } (S,x)P_j(T,y) \\ {}^{(T,y)}R_j & \text{if } (T,y)P_j(S,x). \end{cases}$$

Again, $(S,x)I_j(T,y)$ does not hold since $j \in S$. By monotonicity, $f(R) = f\left(R_j', R_{-j}\right) = f\left(R_i', R_{-i}\right) = f\left(R_i', R_j', R_{-i,j}\right) = (S,x)$.

Claim 1: *$o(i,R) = o(i,(R_i', R_j', R_{-i,j}))$ and $o(j,R) = o(j,(R_i', R_j', R_{-i,j}))$.*

Proof of Claim 1: We only prove that $o(i,R) = o(i,(R_i', R_j'.R_{-i,j}))$ holds (the proof of the case $o(j,R) = o(j,(R_i', R_j', R_{-i,j}))$ is similar). Suppose otherwise. Assume that $o(i,R) \backslash o(i,(R_i', R_j', R_{-i,j})) \neq \varnothing$ (the proof of the other case $o(i,(R_i', R_j', R_{-i,j})) \backslash o(i,R) \neq \varnothing$ is similar and we omit it). Take any alternative $(\widetilde{S}, \widetilde{x}) \in o(i,R) \backslash o(i,(R_i', R_j', R_{-i,j}))$. Since $(\widetilde{S}, \widetilde{x}) \in \tau({}^{(\widetilde{S},\widetilde{x})}R_i)$, $(\widetilde{S}, \widetilde{x}) \in o(i,R) = o(i,({}^{(\widetilde{S},\widetilde{x})}R_i, R_{-i}))$ and, by (3) of Lemma 1, $f({}^{(\widetilde{S},\widetilde{x})}R_i, R_{-i}) = C(o(i,({}^{(\widetilde{S},\widetilde{x})}R_i, R_{-i})), {}^{(\widetilde{S},\widetilde{x})}R_i)$, we have that $f({}^{(\widetilde{S},\widetilde{x})}R_i, R_{-i}) = (\widetilde{S}, \widetilde{x})$. Since $(\widetilde{S}, \widetilde{x}) \notin o(i,(R_i', R_j', R_{-i,j}))$, we have that $f({}^{(\widetilde{S},\widetilde{x})}R_i, R_j', R_{-i,j}) \neq (\widetilde{S}, \widetilde{x})$. Hence, it holds that $L(f({}^{(\widetilde{S},\widetilde{x})}R_i, R_j', R_{-i,j}), {}^{(\widetilde{S},\widetilde{x})}R_i) \subset L(f({}^{(\widetilde{S},\widetilde{x})}R_i, R_j', R_{-i,j}), R_i)$. Since f is monotone, $f({}^{(\widetilde{S},\widetilde{x})}R_i, R_j', R_{-i,j}) = f(R_i, R_j', R_{-ij}) = (S,x)$. We now distinguish between two cases.

Case 1: $(\widetilde{S}, \widetilde{x})P_j'(S,x)$. Then,

$$f({}^{(\widetilde{S},\widetilde{x})}R_i, R_j, R_{-i,j}) = (\widetilde{S}, \widetilde{x})P_j'(S,x) = f({}^{(\widetilde{S},\widetilde{x})}R_i, R_j', R_{-i,j}).$$

Thus, j manipulates f at profile $(^{(\widetilde{S},\widetilde{x})}R_i, R'_j, R_{-i,j})$ via R_j, which is a contradiction.

Case 2: $(S,x)P'_j(\widetilde{S},\widetilde{x})$. By definition of R'_j, $(S,x)P_j(\widetilde{S},\widetilde{x})$. Then,

$$f(^{(\widetilde{S},\widetilde{x})}R_i, R'_j, R_{-i,j}) = (S,x)P_j(\widetilde{S},\widetilde{x}) = f(^{(\widetilde{S},\widetilde{x})}R_i, R_j, R_{-i,j}).$$

Thus, j manipulates f at profile $(^{(\widetilde{S},\widetilde{x})}R_i, R_j, R_{-i,j})$ via R'_j, which is also a contradiction. Since $(\widetilde{S},\widetilde{x})I'_j(S,x)$ is not possible because $i \in S$. we have finished the proof of Claim 1. ■

We now define two new preferences $\tilde{R}_i, \tilde{R}_j$, where

$$\tilde{R}_i = \begin{cases} ^{(T,y)}(_{(U,z)}R_i) & \text{if } (S,x)P_i(U,z) \\ ^{(U,z)}(^{(T,y)}R_i) & \text{if } (U,z)P_i(S,x) \end{cases}$$

and

$$\tilde{R}_j = \begin{cases} ^{(U,z)}(_{(T,y)}R_j) & \text{if } (S,x)P_j(T,y) \\ ^{(T,y)}(^{(U,z)}R_j) & \text{if } (T,y)P_j(S,x). \end{cases}$$

Claim 2: The following two equalities hold.

(i) $f(\tilde{R}_i, R'_j, R_{-ij}) = (T,y)$.

(ii) $f(R'_i, \tilde{R}_j, R_{-ij}) = (U,z)$.

Proof of Claim 2:

(i) Since $(T,y) \in o(i,R)$, by Claim 1, $(T,y) \in o(i,(R'_i,R'_j,R_{-ij}))$. If $(S,x)P_i(U,z)$ we have that $C(o(i,(R'_i,R'_j,R_{-ij})),\tilde{R}_i) = (T,y)$. Since f is strategy-proof, $f(\tilde{R}_i,R'_j,R_{-ij}) = (T,y)$. If $(U,z)P_i(S,x)$ then, $(U,z) \in \tau(\tilde{R}_i)$. Assume first that $(U,z) \in o(i,(R'_i,R'_j,R_{-ij})$. Since f is strategy-proof, $f(\tilde{R}_i,R'_j,R_{-ij}) = (U,z)$. Since $f(R'_i,R'_j,R_{-ij}) = (S,x)$, i manipulates f at profile (R'_i,R'_j,R_{-ij}) via $\tilde{R}_i$, a contradiction. Hence, $(U,z) \notin o(i,(R'_i,R'_j,R_{-ij}))$. Since $(T,y) \in o(i,(R'_i,R'_j,R_{-ij}))$ and f is strategy-proof, $f(\tilde{R}_i,R'_j,R_{-ij}) = (T,y)$.

(ii) We know that $(U,z) \in o(j,R)$. By Claim 1, $(U,z) \in o(j,(R'_i,R'_j,R_{-ij}))$. If $(S,x)P_j(T,y)$ we have that $(U,z) \in C(o(j,(R'_i,R'_j,R_{-ij})),\tilde{R}_j)$. Since f is strategy-proof, $f\left(R'_i,\tilde{R}_j,R_{-ij}\right) = (U,z)$. If $(T,y)P_j(S,x)$ then, $(T,y) \in \tau(\tilde{R}_j)$. Assume first that $(T,y) \in o(j,(R'_i,R'_j,R_{-ij}))$. Since f is strategy-proof, $f(R'_i,\tilde{R}_j,R_{-ij}) = (T,y)$. Since $f(R'_i,R'_j,R_{-ij}) = (S,x)$, j manipulates f at profile (R'_i,R'_j,R_{-ij}) via $\tilde{R}_j$, a contradiction. Hence, $(T,y) \notin o(j,(R'_i,R'_j,R_{-ij}))$. Since $(U,z) \in o(i,(R'_i,R'_j,R_{-ij}))$, and f is strategy-proof, $f(R'_i,\tilde{R}_j,R_{-ij}) = (U,z)$. And this finishes with the proof of Claim 2. ■

We now proceed with the proof of Lemma 3 by considering four different cases:

(1) Assume $(S,x)P_i(U,z)$. Since $f(R'_i,\tilde{R}_j,R_{-ij}) = (U,z)$ by (ii) in Claim 2, we have that $U\left(f\left(R'_i,\tilde{R}_j,R_{-ij}\right),R'_i\right) = A$. Hence, $U(f(R'_i,\tilde{R}_j,R_{-ij}),\tilde{R}_i) \subset U\left(f\left(R'_i,\tilde{R}_j,R_{-ij}\right),R'_i\right)$.

Since f is monotonic,

$$f(\tilde{R}_i, \tilde{R}_j, R_{-ij}) = f(R'_i, \tilde{R}_j, R_{-ij}) = (U, z).$$

(2) Assume $(U, z)P_i(S, x)$. Since $f(R'_i, \tilde{R}_j, R_{-ij}) = (U, z)$ by (ii) in Claim 2, we have that $L(f(R'_i, \tilde{R}_j, R_{-ij}), \tilde{R}_i) = A$. Hence, $L(f(R'_i, \tilde{R}_j, R_{-ij}), R'_i) \subset L(f(R'_i, \tilde{R}_j, R_{-ij}), \tilde{R}_i)$. Since f is monotonic,

$$f(\tilde{R}_i, \tilde{R}_j, R_{-ij}) = f(R'_i, \tilde{R}_j, R_{-ij}) = (U, z).$$

(3) Assume $(S, x)P_j(T, y)$. Since $f(\tilde{R}_i, R'_j, R_{-ij}) = (T, y)$ by (i) in Claim 2, we have that $U(f(\tilde{R}_i, R'_j, R_{-ij}), R'_j) = A$. Hence, $U(f(\tilde{R}_i, R'_j, R_{-ij}), \tilde{R}_j) \subset U(f(\tilde{R}_i, R'_j, R_{-ij}), R'_j)$. Since f is monotonic,

$$f(\tilde{R}_i, \tilde{R}_j, R_{-ij}) = f(\tilde{R}_i, R'_j, R_{-ij}) = (T, y).$$

(4) Assume $(T, y)P_j(S, x)$. Since $f(\tilde{R}_i, R'_j, R_{-ij}) = (T, y)$ by (i) in Claim 2, we have that $L(f(\tilde{R}_i, R'_j, R_{-ij}), \tilde{R}_j) = A$. Hence, $L(f(\tilde{R}_i, R'_j, R_{-ij}), R'_j) \subset L(f(\tilde{R}_i, R'_j, R_{-ij}), \tilde{R}_j)$. Since f is monotonic,

$$f(\tilde{R}_i, \tilde{R}_j, R_{-ij}) = f(\tilde{R}_i, R'_j, R_{-ij}) = (T, y).$$

Thus, in each of the four possible cases $(S, x)P_i(U, z)$ and $(S, x)P_j(T, y)$, $(S, x)P_i(U, z)$ and $(T, y)P_j(S, x)$, $(U, z)P_i(S, x)$ and $(S, x)P_j(T, y)$, and $(U, z)P_i(S, x)$ and $(T, y)P_j(S, x)$, we have that $f(\tilde{R}_i, \tilde{R}_j, R_{-ij}) = (U, z)$ and $f(\tilde{R}_i, \tilde{R}_j, R_{-ij}) = (T, y)$, which is a contradiction. ■

Given $N^* \subset N$ we know, by Lemma 2, that g is dictatorial in the domain $\mathcal{F}^{N^*}$. Let $d(N^*) \in N^*$ be the dictator.

Lemma 4 *Let $R \in \mathcal{R}$ be a profile such that $\tau(R_{d(N^*)}) = (N^*, x)$ for some $x \in X$ and $(N^*, x) \in \bigcap_{j \in N \setminus N^*} \tau(R_j)$. Then, $f(R) = (N^*, x)$.*

Proof of Lemma 4 The proof has four steps.

Step 1: Agent $d(N^*)$ is a dictator in the subdomain of $\mathcal{R}$ induced by the mapping φ. Denote it by $\mathcal{R}^{\varphi} = \{R \in \mathcal{R} \mid R = \left(R_{\varphi^{N^*,i}(\succ_i)}\right)_{i \in N}$ for some $(\succ_i)_{i \in N^*} \in \mathcal{F}^{N^*}\}$. Let $R \in \mathcal{R}^{\varphi}$. By Lemma 2, $f_X(R) = g\left((\succ_i)_{i \in N^*}\right) = x$. By efficiency of f and the definition of $\left(R_{\varphi^{N^*,i}(\succ_i)}\right)_{i \in N}$, $f_N(R) = N^*$.

Step 2. Let $R_{d(N^*)}$ be any preference in the general domain $\mathcal{R}$ such that $\tau\left(R_{d(N^*)}\right) = (N^*, x)$. Hence, for each subprofile $R_{-d(N^*)}$ belonging to the subdomain $\mathcal{R}^{\varphi}_{-d(N^*)}$, $f(R) = (N^*, x)$ follows immediately as the consequence that f is strategy-proof.

Step 3. Let $R \in \mathcal{R}$ be such that $R_{d(N^*)}$ is any preference in $\mathcal{R}$ satisfying $\tau(R_{d(N^*)}) = (N^*, x)$, there exists $i \in N^* \backslash \{d(N^*)\}$ such that R_i is any preference in the general domain $\mathcal{R}_i$, for all $j \in N \backslash \{d(N^*), i\}$, R_j is any preference in the subdomain $\mathcal{R}_j^{\varphi}$. Then, $f(R) = (N^*, x)$. Consider any preference R_i' in the subdomain $\mathcal{R}_i^{\varphi}$. By Step 2, for all $y \in X$, $(N^*, y) \in o(d(N^*), (R_i', R_{-i}))$. Since $|o(d(N^*), (R_i', R_{-i}))| \geq 3$, by Lemma 3, $|o(i, (R_i', R_{-i}))| = 1$ holds. Since $(N^*, x) = f(R)$ and $o(i, (R_i', R_{-i},)) = o(i, R)$, $(N^*, x) \in o(i, (R_i', R_{-i}))$ and hence, $o(i, (R_i', R_{-i})) = (N^*, x)$. Thus, $f(R) = (N^*, x)$.

Step 4. Applying successively the arguments of Step 3 we obtain that for all $R \in \mathcal{R}$ satisfying (i) $R_{d(N^*)}$ is any preference in $\mathcal{R}$ satisfying $\tau(R_{d(N^*)}) = (N^*, x)$, (ii) for all $i \in N^* \backslash \{d(N^*)\}$, R_i is any preference in the general domain $\mathcal{R}$, and (iii) for all $j \in N \backslash N^*$, R_j is any preference in the subdomain $\mathcal{R}_j^{\varphi}$, we have that $f(R) = (N^*, x)$. ■

Lemma 5 *Assume $N' \subsetneq N'' \subset N$ are such that $d(N'') \in N'$. Then, $d(N') = d(N'')$.*

Proof of Lemma 5 Suppose not. Let $x \in X$ and consider the profile $R \in \mathcal{R}$ where (i) $R_{d(N'')}$ satisfies $\tau\left(R_{d(N'')}\right) = (N'', x)$, (ii) $R_{d(N')}$ satisfies $\tau\left(R_{d(N')}\right) = (N', x)$ and (iii) for each $i \in N \backslash \{d(N'), d(N'')\}$, R_i is any preference in the subdomain $\mathcal{R}_i^{\varphi}$ for $N^* = \{d(N'), d(N'')\}$. By Lemma 4, with $N^* = N''$, $f(R) = (N'', x)$. By Lemma 4 again, with $N^* = N'$, $f(R) = (N', x)$, which is a contradiction. ■

Proof of Theorem 1 Let $\pi \in \Pi$ and $x \in X$ be given. It is easy to show that the serial dictator rule $f^{\pi,x}$ is strategy-proof, unanimous and nonbossy. To prove the other implication, assume $f : \mathcal{R} \to A$ is a strategy-proof, unanimous and nonbossy rule. We will identify from f a permutation of agents $\pi \in \Pi$ and an alternative $x \in X$ such that $f = f^{\pi,x}$. We first define π recursively by setting $\pi_1 = d(N)$ and, for all $i = 2, ..., n$, $\pi_i = d(N \backslash \{\pi_1, ..., \pi_{i-1}\})$. To identify $x \in X$, let $R \in \mathcal{R}$ be such that for each $i \in N$, $\tau(R_i) = [\varnothing]_i$. Thus,

$$\bigcap_{i \in N} \tau(R_i) = \{(\varnothing, x') \in A \mid x' \in X\}.$$

By unanimity, $f(R) \in \bigcap_{i \in N} \tau(R_i)$. Set $x = f_X(R)$. We now prove that $f = f^{\pi,x}$. Let $R \in \mathcal{R}$ be an arbitrary profile. Two cases are possible:

Case 1. $|\tau(R_{\pi_1})| = 1$. Thus, $\tau(R_{\pi_1}) = (S_1, x_1)$ and $\pi_1 \in S_1$. By definition, $f^{\pi,x}(R) = (S_1, x_1)$. For each $i \in N \backslash S_1$, let R_i' be any preference in the subdomain $\mathcal{R}_i^{\varphi}$ induced by φ when $N^* = S_1$. Since $\pi_1 \in S_1$, by Lemmata 4 and 5, $f(R_{S_1}, R'_{-S_1}) = (S_1, x_1)$. Let $i \in N \backslash S_1$, By Lemma 4, $(S_1 \cup \{i\}, y) \in o(\pi_1, (R_{S_1 \cup \{i\}}, R'_{-(S_1 \cup \{i\})}))$ for all $y \in X$. By Lemma

3, $\left|o(i,(R_{S_1\cup\{i\}},R'_{-(S_1\cup\{i\})}))\right| = 1$. Since $o(i,(R_{S_1\cup\{i\}},R'_{-(S_1\cup\{i\})})) = o(i,(R_{S_1},R'_{-S_1}))$, we deduce that $\left|o(i,(R_{S_1},R'_{-S_1}))\right| = 1$. Since $f(R_{S_1},R'_{-S_1}) = (S_1,x_1)$, $o(i,(R_{S_1},R'_{-S_1})) = (S_1,x_1)$. Hence, $f(R_{S_1\cup\{i\}},R'_{-(S_1\cup\{i\})}) = (S_1,x_1)$. Similarly, we can also prove easily that $f(R_{S_1\cup\{i,j\}},R'_{-(S_1\cup\{i,j\})}) = (S_1,x_1)$ holds when $j \in N\backslash(S_1\cup\{i\})$. Repeating this process for the rest of the agents in $N\backslash S_1$, we obtain that $f(R) = (S_1,x_1)$. Hence, $f^{\pi,x}(R) = f(R)$.

Case 2. $|\tau(R_{\pi_1})| > 1$. Thus, $\tau(R_{\pi_1}) = [\varnothing]_{\pi_1}$. We consider two subcases separately.

Case 2.1. $|C(\tau(R_{\pi_1}),R_{\pi_2})| = 1$. Thus, $C(\tau(R_{\pi_1}),R_{\pi_2}) = (S_2,x_2)$ and $\pi_2 \in S_2$. It is immediate to see that $f^{\pi,x}(R) = (S_2,x_2)$. We now prove that $f(R) = (S_2,x_2)$. For each $i \in N\backslash S_1$, let R'_i be any preference in the subdomain $\mathcal{R}_i^{\varphi}$ induced by φ when $N^* = S_2$. Take $R'_{\pi_1} = R_{\pi_1}$ (remember that R_{π_1} belongs to the subdomain $\mathcal{R}_{\pi_i}^{\varphi}$). Using arguments similar to those used in Case 1, we can show that $f(R) = (S_2,x_2)$.

Case 2.2. $|C(\tau(R_{\pi_1}),R_{\pi_2})| > 1$. Thus,

$$C(\tau(R_{\pi_1}),R_{\pi_2}) = \{(S,y) \in A \mid \pi_1 \notin S,\ \pi_2 \notin S \text{ and } y \in X\}.$$

We would consider again two subcases separately depending on $|C(C(\tau(R_{\pi_1}),R_{\pi_2}),R_{\pi_3})|$. Continuing with this procedure, at the end we would reach agent n and we would need to consider two subcases separately depending on $|C(A_n,R_{\pi_n})|$, where

$$A_n = \{(\{n\},y) \in A \mid y \in X\} \cup \{(\varnothing,y) \in A \mid y \in X\}.$$

If $|C(A_n,R_{\pi_n})| = 1$ then $C(A_n,R_{\pi_n}) = (\{n\},x_n)$. Thus, $f^{\pi,x}(R) = (\{n\},x_n)$. Using arguments similar to those used above we can show that $f(R) = (\{n\},x_n)$.

If $|C(A_n,R_{\pi_n})| > 1$ then $C(A_n,R_{\pi_n}) = \{(\varnothing,y) \in A \mid y \in X\}$. Then, $f^{\pi,x}(R) = (\varnothing,x)$. By definition of x, $f(R) = (\varnothing,x)$. ■

The three properties used in the characterization of Theorem 1 are independent.

The Approval voting rule $f^{AV,\rho}$ defined in Section 5 satisfies nonbossiness and unanimity but fails strategy-proofness.

Any constant rule satisfies strategy-proofness and nonbossiness but fails unanimity.

Let $x,y \in X$ with $x \neq y$. We define

$$f(R) = \begin{cases} f^{\pi,x} & \text{if } \tau(R_{\pi_1}) = [\varnothing]_i \text{ and } (1,x)\,P_1\,(1,y) \\ f^{\pi,y} & \text{otherwise.} \end{cases}$$

It is easy to see that f satisfies strategy-proofness and unanimity but fails nonbossiness.